Fourier-type density estimation in a tomography problem

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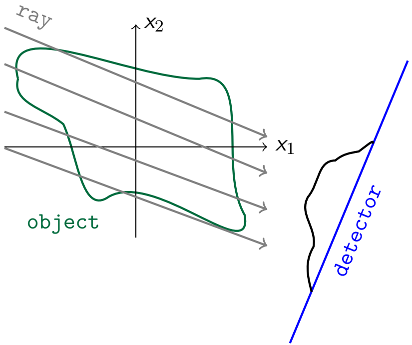
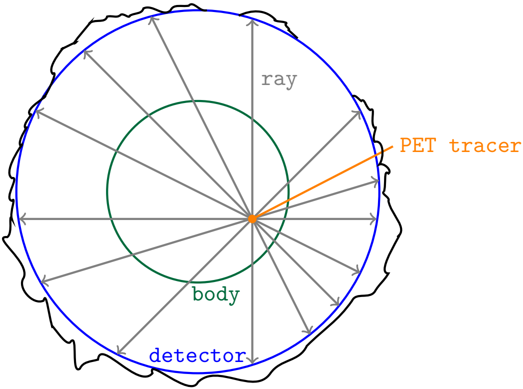
Introduction to the tomography: Physical background

- reconstruction of the internal structure of an object of interest
- computerized tomography: fixed positioned thin X-rays to get a cross-section
  $\rightsquigarrow$ interpret straight line as the two-dimensional Radon transform



Introduction to the tomography: Physical background

- reconstruction of the internal structure of an object of interest
- positron emission tomography: reconstruction of the unknown position on specific lines



Introduction to the tomography: Statistical framework

- computerized tomography: $Y = \mathcal{R}(X) + \varepsilon$
- positron emission tomography: $X \sim c \cdot \mathcal{R}[f]$

Radon transform of f: $\mathcal{R}[f]$
[BHP14]

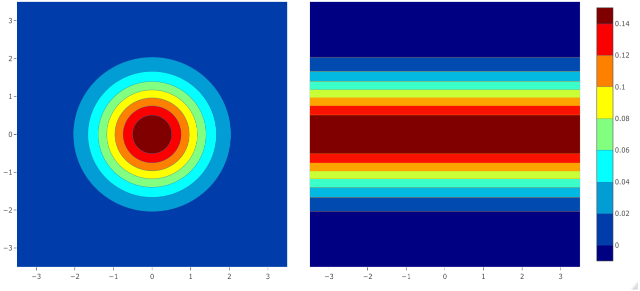
Radon transform $\mathcal{R}[f]$ of $f : \mathbb{R}^d \to \mathbb{R}$, $f \in \mathbb{L}^1\left(\mathbb{R}^d\right)$

$$\mathcal{R}[f](s,u) := \int_{\left\{v \in \mathbb{R}^d : \langle v, s\rangle = u\right\}} f(v)\, \mathrm{d}v,$$

where $u \in \mathbb{R}$, $s \in \mathbb{S}^{d-1}$, $\mathbb{S}^{d-1} := \left\{v \in \mathbb{R}^d : |v|_d = 1\right\}$ is the unit sphere in $\mathbb{R}^d$ and the integration is over the hyperplane $\left\{v \in \mathbb{R}^d : \langle v, s\rangle = u\right\}$ in $\mathbb{R}^d$

Radon transform of the multivariate normal distribution

$$f(x) = (2\pi\sigma^2)^{-d/2} \exp\left(-\left|x-\mu\right|_d^2 / (2\sigma^2)\right),\ x, \mu \in \mathbb{R}^d,$$
$$\sigma \in (0,\infty)$$
$$\Rightarrow \mathcal{R}[f](s,u) = (2\pi\sigma^2)^{-1/2} \exp\left(-|u - \langle \mu, s\rangle|^2/(2\sigma^2)\right),$$
$$(s,u) \in \mathbb{S}^{d-1} \times \mathbb{R}$$



An approximation of the bivariate standard normal distribution (left) and its sinogram (right).

Model introduction

$(S_1, U_1), \ldots, (S_n, U_n)$ i.i.d. observations from a probability density $\rho_d^{-1}\mathcal{R}[f](s,u)$ on $\mathcal{Z} := \mathbb{S}^{d-1} \times \mathbb{R}$, where $\rho_d = 2\pi^{d/2}/\Gamma(d/2)$

$\Rightarrow$ kernel density estimator of $f(x)$ at a fixed point $x \in \mathbb{R}^d$:

$$\widehat{f}_m(x) := \frac{1}{n}\sum_{i=1}^{n} K_m\left(\langle S_i, x\rangle - U_i\right),$$

where $m > 0$ and, for $u \in \mathbb{R}$,

$$K_m(u) := \mathcal{F}_1^{-1}\left[\mathbb{1}_{[-m,m]}(2\pi)^{1-d}\rho_d|\cdot|^{d-1}/2\right](u)$$
$$= \rho_d(2\pi)^{-d}\int_0^m r^{d-1}\cos(ur)\,\mathrm{d}r.$$

[Cav00]

Local measure of accuracy: pointwise quadratic risk

Assumption: $\|\mathcal{F}_d[f]\|_{\mathbb{L}^1(\mathbb{R}^d)} < \infty$

$\Rightarrow$ point evaluation

$$f(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} \exp(-i\langle\omega, x\rangle)\mathcal{F}_d[f](\omega)\, \mathrm{d}\lambda^d(\omega)$$

is well-defined for all $x \in \mathbb{R}^d$ (using the inversion formula of the Fourier transform)

Mean squared error: upper bound

Theorem (Upper bound over $\mathcal{W}^s(L)$):

Let $x \in \mathbb{R}^d$, $s > d/2$ and $L > 0$. Then for spectral cut-off parameters $m_n := n^{1/(2s+d-1)}$ it holds

$$\sup_{f \in \mathcal{W}^s(L)} \mathbb{E}_f \left[\left| f(x) - \widehat{f}_{m_n}(x) \right|^2 \right] \leq C(L, s, d) n^{-\frac{2s-d}{2s+d-1}}.$$

$\mathcal{W}^s(L) := \left\{ f \in \mathbb{L}^2(\mathbb{R}^d) : \int_{\mathbb{R}^d} (1 + |t|_d^2)^s |\mathcal{F}_d[f](t)|^2 \, \mathrm{d}\lambda^d(t) \leq L \right\}$, $s, L > 0$

In [Cav00], different regularity spaces have been considered (exponential decay of Fourier transform).

Mean squared error: lower bound

Theorem (Minimax lower bound over $\mathcal{W}^s(L)$)

Let $x_0 \in \mathbb{R}^d, L, s > 0$. Then, there exist constants $L_{s,d}, C(L,s,d) > 0$, such that for all $L > L_{s,d}$ it holds that

$$\inf_{\widehat{f}(x_0)} \sup_{f \in \mathcal{W}^s(L)} \mathbb{E}_f\left[\left|\widehat{f}(x_0) - f(x_0)\right|^2\right] \geq C(L,s,d) n^{-\frac{2s-d}{2s+d-1}},$$

where $\widehat{f}(x_0)$ is an estimator based on an i.i.d. sample $(S_1, U_1), \ldots, (S_n, U_n)$.

$\mathcal{W}^s(L) := \left\{ f \in \mathbb{L}^2(\mathbb{R}^d) : \int_{\mathbb{R}^d} \left(1 + |t|_d^2\right)^s |\mathcal{F}_d[f](t)|^2 \, \mathrm{d}\lambda^d(t) \leq L \right\}$, $s, L > 0$

In the paper, different regularity spaces have been considered (exponential decay of Fourier transform) and an alternative proof structure (here: oriented on [Tsy08]).

Data-driven choice: Goldenshluger–Lepski method, pointwise version

Theorem

Let $\mathcal{M}_n := \left\{ m \in \mathbb{N} : m \leq n^{1/(2d-1)} \right\}$. Let $f \in \mathbb{L}^2\left(\mathbb{R}^d\right)$, $\mathcal{F}_d[f] \in \mathbb{L}^1\left(\mathbb{R}^d\right)$. Then for $\chi_0 \geq 48$ it holds that

$$\mathbb{E}_f\left[\left|\widehat{f}_{\widehat{m}(x_0)}(x_0) - f(x_0)\right|^2\right] \leq C_1 \inf_{m \in \mathcal{M}_n}\left(\left\| \mathbb{1}_{\mathrm{B}^c_m(0)} \mathcal{F}_d[f] \right\|^2_{\mathbb{L}^1(\mathbb{R}^d)} + V(m) \right) + \frac{C_2}{n},$$

where $x_0 \in \mathbb{R}^d$, $V(m) := \chi_0 C_d \left(1 + \|\mathcal{F}_d[f]\|_{\mathbb{L}^1(\mathbb{R}^d)}\right) m^{2d-1} \log(n) n^{-1}$.

$C_1 > 0$ depending on d and $C_2 > 0$ depending on d and $\|\mathcal{F}_d[f]\|_{\mathbb{L}^1(\mathbb{R}^d)}$

Data-driven choice: Goldenshluger–Lepski method, pointwise version

Proof (Sketch of the proof)

Based on the Goldenshluger–Lepski proof for the Fourier estimation.

2 main steps:

- elementary steps to find a controllable upper bound for the risk
- Bernstein inequality
  Let $T_1, \ldots, T_n$ be i.i.d. random variables, and we define $S_n(T) := \sum_{i=1}^{n} (T_i - \mathbb{E}[T_i])$. Then, for any $\eta > 0$, we get $\mathbb{P}(|S_n(T) - \mathbb{E}[S_n(T)]| \geq n\eta) \leq 2\max\left(\exp\left(-\frac{n\eta^2}{4v^2}\right), \exp\left(-\frac{n\eta}{4b}\right)\right)$, where $\mathbb{V}\mathrm{ar}(T_1) \leq v^2$ and $|T_1| \leq b$, for some positive constants v and b.

□

[Com17]

Data-driven choice: Goldenshluger–Lepski method, pointwise version

Corollary

Let $L > 0$, $s > d/2$. Then for any $\chi_0 \geq 48$

$$\sup_{f \in \mathcal{W}^s(L)} \mathbb{E}_f \left[\left| \widehat{f}_{\widehat{m}(x_0)}(x_0) - f(x_0) \right|^2 \right] \leq C(L, s, d, \chi_0) \left(\frac{n}{\log(n)} \right)^{-\frac{2s-d}{2s+d-1}}.$$

- Optimal rate up to a $(\log(n))$-term.

$\mathcal{W}^s(L) := \left\{ f \in \mathbb{L}^2(\mathbb{R}^d) : \int_{\mathbb{R}^d} \left(1 + |t|_d^2\right)^s |\mathcal{F}_d[f](t)|^2 \, d\lambda^d(t) \leq L \right\}, s, L > 0$

Lower bound for data-driven estimators

Theorem

Let $x \in \mathbb{R}^d$, $s > d/2$. Then, there exists $L_{s,d} > 0$ such that for all $L \geq L_{s,d}$ holds:

If a sequence of estimators $\{\widehat{f}_n(x)\}_{n \in \mathbb{N}}$ of $f(x)$ based on the data $(S_1, U_1), \ldots, (S_n, U_n)$ satisfies

$$\sup_{n \in \mathbb{N}} \sup_{f \in \mathcal{W}^s(L)} \mathbb{E}_{\mathcal{R}[f]} \left[\left| \widehat{f}_n(x) - f(x) \right|^2 \right] n^{\frac{2s-d}{2s+d-1}} \leq \mathfrak{C},$$

then for any $s' \in (d/2, s)$ there exists $\mathfrak{c} > 0$ such that

$$\liminf_{n \to \infty} \sup_{f \in \mathcal{W}^{s'}(L)} \mathbb{E}_{\mathcal{R}[f]} \left[\left| \widehat{f}_n(x) - f(x) \right|^2 \right] \left(\frac{n}{\log(n)} \right)^{\frac{2s'-d}{2s'+d-1}} \geq \mathfrak{c}.$$

$\mathcal{W}^s(L) := \left\{ f \in \mathbb{L}^2(\mathbb{R}^d) : \int_{\mathbb{R}^d} (1 + |t|_d^2)^s |\mathcal{F}_d[f](t)|^2 \, \mathrm{d}\lambda^d(t) \leq L \right\}$, $s, L > 0$

$\mathfrak{c} = \mathfrak{c}(\mathfrak{C}, s, s', d) > 0$

Thank you for your attention!

References

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[Tsy08] Aleksandr B. Tsybakov. *Introduction to nonparametric estimation*. Springer Publishing Company, Incorporated, 2008.