Robust Score Matching

Using the geometric median

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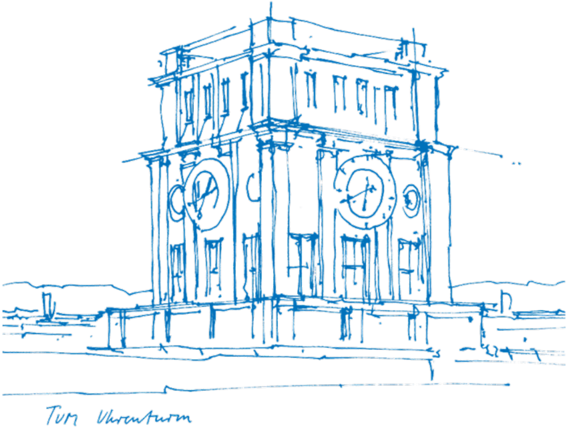
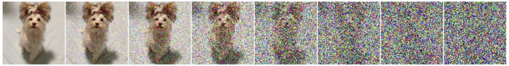


Outline

1. Score Matching using the Geometric Median (AISTATS 2025)
2. Some phenomena in high dimensions (arXiv:2508.12926)

Setup

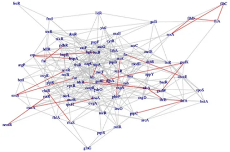
Score Matching



$$\nabla \log(c \cdot p(x)) = \nabla \log(p(x))$$

Robustness

Exponential family/
Graphical model



Exponential family score matching

Density model: $p(\mathbf{x} \mid \boldsymbol{\theta}) = \exp\left(\boldsymbol{\theta}^\top \mathbf{t}(\mathbf{x}) - a(\boldsymbol{\theta}) + b(\mathbf{x})\right)$ with $\mathbf{x} \in \mathcal{X}$ and parameter $\boldsymbol{\theta} \in \mathbb{R}^r$.

$$\boxed{\boldsymbol{\theta}_0} = \operatorname{argmin}_{\boldsymbol{\theta}} \ \frac{1}{2} \mathrm{E}_{\mathbf{x}\sim p(\cdot|\boldsymbol{\theta}_0)}\left[\, \|\nabla_{\mathbf{x}} \log(p(\mathbf{x} \mid \boldsymbol{\theta})) - \nabla_{\mathbf{x}} \log(p(\mathbf{x} \mid \boldsymbol{\theta}_0))\|^2 \,\right]$$

$$= \operatorname{argmin}_{\boldsymbol{\theta}} \ \frac{1}{2} \mathrm{E}_{\boldsymbol{\theta}_0}\left[\, \|\nabla_{\mathbf{x}} \log(p(\mathbf{x} \mid \boldsymbol{\theta}))\|^2 \,\right] + \mathrm{E}_{\boldsymbol{\theta}_0}\left[\, \Delta_{\mathbf{x}} \log(p(\mathbf{x} \mid \boldsymbol{\theta})) \,\right]$$

$$\boxed{=: \operatorname{argmin}_{\boldsymbol{\theta}} \ \tfrac{1}{2}\boldsymbol{\theta}^T \ \mathrm{E}_{\boldsymbol{\theta}_0}[\,\boldsymbol{\Gamma}(\mathbf{x})\,]\,\boldsymbol{\theta} + \boldsymbol{\theta}^T \ \mathrm{E}_{\boldsymbol{\theta}_0}[\,\mathbf{g}(\mathbf{x})\,]}$$

for some $\boldsymbol{\Gamma}(\mathbf{x}) \in \mathbb{R}^{r\times r}$ and $\mathbf{g}(\mathbf{x}) \in \mathbb{R}^r$.

Example (Square root graphical model)

Let $p(\mathbf{x} \mid \boldsymbol{\theta}) \propto \exp\left(-\sqrt{\mathbf{x}}^T \boldsymbol{\Theta} \sqrt{\mathbf{x}}\right)$, $\mathbf{x} \in \mathbb{R}^m_+$. Then,

$$\boldsymbol{\Gamma}(\mathbf{x}) = \operatorname{diag}(1/\mathbf{x}) \otimes \sqrt{\mathbf{x}}\sqrt{\mathbf{x}}^T \in \mathbb{R}^{m^2\times m^2}$$

Robust score matching - considerations

Standard score matching estimator $\hat{\boldsymbol{\theta}} := \mathrm{argmin}_{\boldsymbol{\theta}} \; \frac{1}{2}\boldsymbol{\theta}^T \bar{\boldsymbol{\Gamma}} \boldsymbol{\theta} + \boldsymbol{\theta}^T \bar{\mathbf{g}}$ with $\bar{\boldsymbol{\Gamma}} := \frac{1}{n}\sum_{i=1}^{n} \boldsymbol{\Gamma}(\mathbf{x}_i)$.
Estimator not robust, e.g. against $\|\boldsymbol{\Gamma}(\mathbf{x}_1)\| \to \infty$.

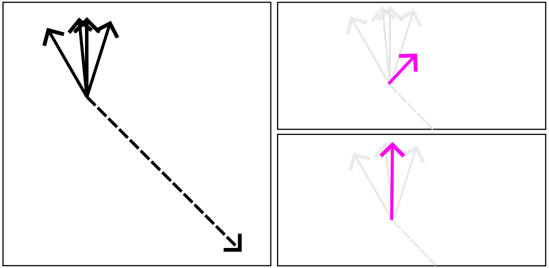
Desiderata for alternatives $\hat{\boldsymbol{\Gamma}}, \hat{\boldsymbol{g}}$

- Robust against arbitrary contamination of a few observations $\mathbf{x}_i$ (i.e. rowwise contamination)
- Few assumptions on $\boldsymbol{\Gamma}, \boldsymbol{g}$ required
- Computable in high dimensions
- $\hat{\boldsymbol{\Gamma}}$ should be positive (semi)definite

The geometric median

Definition: $\mathrm{Med}(\,\mathbf{x}_1, \dots, \mathbf{x}_n\,) := \mathrm{argmin}_{\mathbf{m}} \; \frac{1}{n}\sum_{i=1}^{n} \|\mathbf{x}_i - \mathbf{m}\|_2$

Illustration:



Mean

Geometric median

- Robustness and computation well studied in the literature
- Geometric median exists & unique under mild assumptions
- $\mathrm{Med}(\,\mathbf{x}_1, \dots, \mathbf{x}_n\,) \in$ ConvHull$(\mathbf{x}_1, \dots, \mathbf{x}_n)$

  $\Rightarrow$ When $\mathbf{\Gamma}(\mathbf{x}_i)$ is pos. sem. def., then so is $\mathrm{Med}(\mathbf{\Gamma}(\mathbf{x}_1), \dots, \mathbf{\Gamma}(\mathbf{x}_n))$

Robust Score Matching

We consider $\hat{\boldsymbol{\theta}} := \mathrm{argmin}_{\boldsymbol{\theta}} \ \frac{1}{2}\boldsymbol{\theta}^T \hat{\boldsymbol{\Gamma}} \boldsymbol{\theta} + \boldsymbol{\theta}^T \hat{\mathbf{g}}$ with $\hat{\boldsymbol{\Gamma}} := \mathrm{Med}(\boldsymbol{\Gamma}(\mathbf{x}_1), \ldots, \boldsymbol{\Gamma}(\mathbf{x}_n))$.

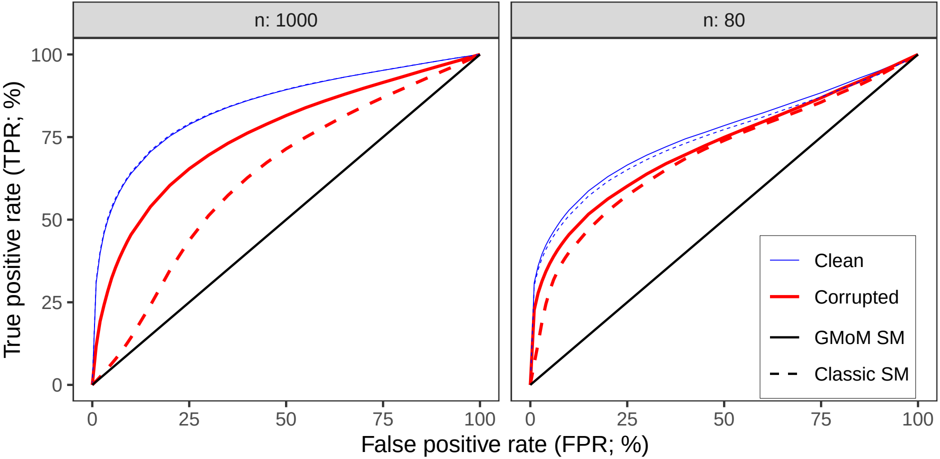
Additionally, we allow for

- ℓ_1-regularization
- support constraints (e.g. $\mathbf{x} > 0$; *generalized* score matching)
- median-of-means to balance bias with robustness.

For pairwise interaction models, we prove edge recovery under standard Lasso-assumptions, even if a fraction of the n observations is arbitrarily contaminated.

Simulation results for graphical models

Task: find edges in square root graphical model. Below: ROC curves (ℓ_1-penalty varies)



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Bias in high dimensions

Observation from robust score matching:

On uncorrupted data, geometric median-based $\hat{\mathbf{\Gamma}}$ surprisingly close to sample mean $\bar{\mathbf{\Gamma}}$ when

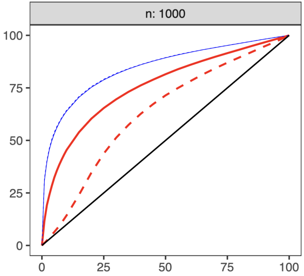
- sample size n is large *and*
- number of observed variables is high.

Large-n-limit: Given random variable $\mathbf{x} \in \mathbb{R}^p$, define

- $\mathbf{m} := \mathrm{Med}(\mathbf{x}) := \mathrm{argmin}_{\tilde{\mathbf{m}}} \mathrm{E}[\, \|\mathbf{x} - \tilde{\mathbf{m}}\| \,]$
- $\boldsymbol{\mu} := \mathrm{E}[\, \mathbf{x} \,]$.

Research Question: Quantify $\|\mathbf{m} - \boldsymbol{\mu}\|$ as $p \to \infty$.

From last slide:



A simple example

Consider $\mathbf{x} \in \mathbb{R}^p$ with x_i iid. Exp(1). Geometric median $\mathbf{m}$ exists & unique.

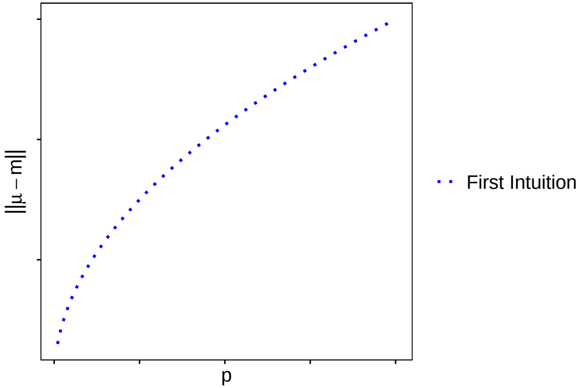
How does $\|\mathbf{m} - \boldsymbol{\mu}\|$ scale with p?

A simple example

Consider $\mathbf{x} \in \mathbb{R}^p$ with x_i iid. Exp(1). Geometric median $\mathbf{m}$ exists & unique.

How does $\|\mathbf{m} - \boldsymbol{\mu}\|$ scale with p?

First intuition: Componentwise median $\tilde{\mathbf{m}}$ has $\|\tilde{\mathbf{m}} - \boldsymbol{\mu}\| = |\log(2) - 1| \cdot \sqrt{p}$



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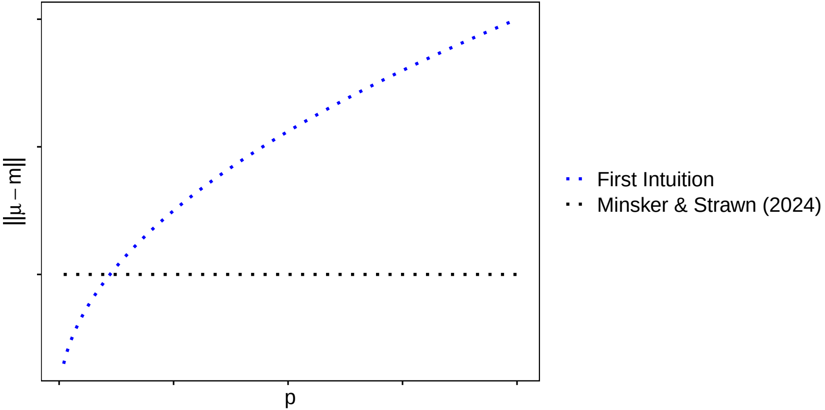

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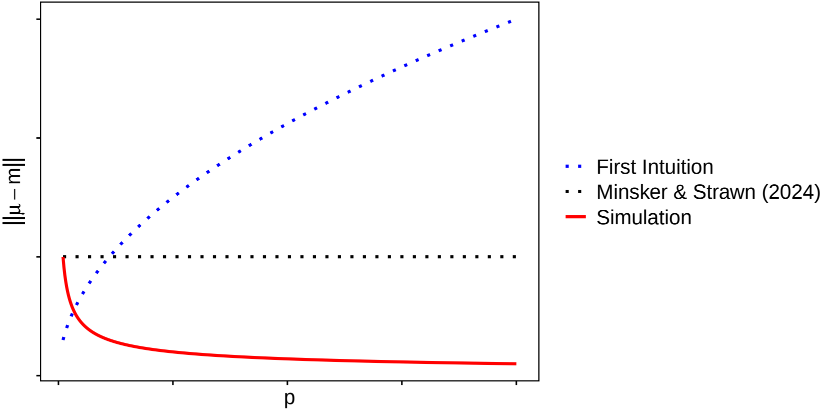

Theoretical insights

arXiv > math > arXiv:2508.12926

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Mathematics > Statistics Theory

[Submitted on 18 Aug 2025]

On the distance between mean and geometric median in high dimensions

Richard Schwank, Mathias Drton

The geometric median, a notion of center for multivariate distributions, has gained recent attention in robust statistics and machine learning. Although conceptually distinct from the mean (i.e., expectation), we demonstrate that both are very close in high dimensions when the dependence between the distribution components is suitably controlled. Concretely, we find an upper bound on the distance that vanishes with the dimension asymptotically, and derive a rate-matching first order expansion of the geometric median components. Simulations illustrate and confirm our results.

Subjects: **Statistics Theory (math.ST)**

Cite as: arXiv:2508.12926 **[math.ST]**

(or arXiv:2508.12926v1 **[math.ST]** for this version)

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We prove $\|\boldsymbol{\mu} - \mathbf{m}\| = \mathcal{O}\left(\frac{1}{\sqrt{p}}\right)$ when components are M-dependent.

Discussion & Implications

- Similarities with (Brown 1983): sample geometric median becomes efficient for isotropic Gaussian as $p \to \infty$
- If M-dependence $\approx$ edge sparsity, can explain Robust SM's small bias on uncorrupted data
- Consistency check: rowwise contaminated observations generally *not* M-dependent
- Note: certain *cellwise* contaminated observations are M-dependent. Indeed, geometric median *not* robust (Raymaekers and Rousseeuw 2024)
- “Robustness" depends on contamination assumption!

Thank you!

- Schwank, R., McCormack, A., Drton, M. (2025). Robust score matching, AISTATS 2025.
- Schwank, R., Drton, M. (2025). On the distance between mean and geometric median in high dimensions. 10.48550/arXiv.2508.12926.

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References

Brown, B. M. (Sept. 1983). "Statistical Uses of the Spatial Median". In: *Journal of the Royal Statistical Society: Series B (Methodological)* 45.1, pp. 25–30. ISSN: 2517-6161. DOI: 10.1111/j.2517-6161.1983.tb01226.x.

Minsker, Stanislav and Nate Strawn (July 2023). "The Geometric Median and Applications to Robust Mean Estimation". In: DOI: 10.48550/ARXIV.2307.03111. arXiv: 2307.03111 [math.ST].

Raymaekers, Jakob and Peter J. Rousseeuw (Feb. 2024). "Challenges of cellwise outliers". In: *Econometrics and Statistics*. ISSN: 2452-3062. DOI: 10.1016/j.ecosta.2024.02.002.